Evaluating and expressing uncertainty in complex *S*-parameter measurements

Nick M. Ridler and Martin J. Salter
Centre for Electromagnetic and Time Metrology
National Physical Laboratory, Teddington, Middlesex, TW11 0LW, United Kingdom

Abstract

This paper presents methods for evaluating and expressing the uncertainty associated with complex *S*-parameter measurements. The methods are based on internationally recommended guidelines [1] with extensions to accommodate the complex nature of the measurands. The treatment of measurements of both one-port and multi-port devices is presented.

1. Introduction

Nowadays, there are available several national [2,3] and international [1] documents giving guidance on evaluating and expressing the uncertainty of measurement results. However, the emphasis in these documents is on dealing with scalar measurement quantities. Therefore, this information is of limited use when dealing with vector measurement quantities, such as complex reflection and transmission coefficients (*i.e. S*-parameters).

There are several important issues which need to be addressed when dealing with complex-valued measurement data. For example, what form should be used to express the complex quantity? This could either be in terms of the real and imaginary components of the complex value or its magnitude and phase. Also, since a complex number consists of two components, any correlation between these components needs to be considered. This is particularly true when a detailed description of the uncertainty of measurement in the complex quantity is required. This also leads to the need to apply suitable coverage factors to obtain uncertainty intervals at a specified level of confidence.

This paper addresses the issues outlined above and presents what is effectively an extension to the principles given in the current guidance documents [1-3] appropriate for complex-valued quantities. Consideration is given to *S*-parameter measurements of both one-port and multi-port devices.

2. Parameter representation – the difficulties

There are two obvious ways of representing a complex measurand: either in terms of its real and imaginary components, or its magnitude and phase. In the majority of engineering applications, the magnitude and phase representation is generally preferred. This is because this representation bears a direct relationship to physical phenomena affecting the measurement. For example, phase relates directly to electrical path length, and magnitude relates directly to signal loss (or signal absorbed). The same cannot be said for the representation of a complex measurand in terms of its real and imaginary components.

However, when evaluating the uncertainty in a complex measurand, there are additional considerations which need to be taken into account. These relate mainly to the use of statistics to evaluate the size of the random errors in the measurement process (the so-called type-A evaluation of uncertainty). In particular, applying statistics to analyse data involves performing a number of arithmetic operations on the data (*e.g.* addition, subtraction, etc). When we examine the scales used to depict the different representations of a complex measurand, we find the following: the real and imaginary axes in the complex plane extend, in principle, to infinity in both directions (*i.e.* $\pm \infty$). This is the same as the scale used to depict all real numbers which are routinely subject to the arithmetic operations mentioned above. However, the scales used to represent magnitude and phase each possess a peculiarity. In the case of magnitude, there is a lower bound of zero below which values cannot exist (it is nonsense to talk about a vector with negative magnitude), and phase is represented on a scale which is cyclical in nature (usually either from -180º to +180º or from 0º to 360º).

The peculiarities in both these scales can cause problems when using statistics to summarise the complex data. For example, when averaging repeat measurements of low magnitude *S*-parameters, a systematic error can be introduced as a consequence of the averaging process if the magnitude values themselves are averaged. This method of averaging can be expressed mathematically as:

$$\overline{|S|} = \frac{1}{n}\sum_{i=1}^{n}|S|_i \qquad (1)$$

where $\overline{|S|}$ is the mean magnitude of an arbitrary *S*-parameter and $|S|_i$ represents the magnitude of the *i*th of *n* measurements of *S*. Now, let us consider a notionally perfect matched-load termination ($|S_{11}| = 0$). Every measurement of $|S_{11}|$ will always be greater than zero due to random errors in the measurement process. If the repeat measurements of S_{11} (complex) were displayed in the complex plane, we can assume that they would appear as a two-dimensional, *i.e.* bivariate, normal distribution centred on the origin of the complex plane. However, when the distribution of $|S_{11}|$ is plotted, instead of the expected scenario of a normal distribution centred on the notionally 'true' value[1] (*i.e.* $|S_{11}| = 0$), we find the distribution is skewed (*i.e.* a Rayleigh distribution[2]) with the peak at a value greater that the 'true' value! Hence, $\overline{|S|} > |S_{true}|$ when $|S_{true}|$ is small, *i.e.* $\overline{|S|}$ is a biased estimator of $|S_{true}|$.

A similar problem occurs when estimating the standard uncertainty in $\overline{|S|}$, $s(\overline{|S|})$, using:

$$s(\overline{|S|}) = \sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}(|S|_i - \overline{|S|})^2} \qquad (2)$$

In this case, the calculation of the standard uncertainty under-estimates the 'true' underlying standard uncertainty. Both these effects have been analysed in detail in [4], and an alternative treatment of the same effect is given in [5].

Calculations performed on phase measurements also present difficulties for conventional arithmetic. The cyclical nature of the phase scale requires a form of 'clock' (or modular)

[1] Ignoring any effects caused by systematic errors in the measurement process.

[2] An analogy here is when a skilled darts player (*i.e.* one with a good aim, and without any systematic bias) throws darts at a dartboard aiming at the point centre (a point with zero area). If we measure the distance from the point centre to the position where each dart has landed, and plot these distances, we arrive at the same distribution, which is truncated at zero.

arithmetic to be used when performing calculations. For example, averaging the phase values +179º and -177º using conventional arithmetic leads to a result of +1º, since:

$$\frac{+179° + -177°}{2} = +1°$$

whereas intuitively, one would expect the result to be -179º (on a scale ranging from -180º to +180º).

3. Parameter representation – recommendations

Considering the difficulties that can be encountered when analysing complex-valued measurement data (presented in the previous section), we make the following recommendations:

1) ensure all complex-valued measurements are expressed in terms of real and imaginary components before performing any arithmetic operations (*e.g.* when applying statistical techniques to analyse complex-valued data).

2) Calculate the mean of the real component, $\overline{\mathrm{Re}(S)}$, of the complex quantity, *S*, using:

$$\overline{\mathrm{Re}(S)} = \frac{1}{n}\sum_{i=1}^{n}\mathrm{Re}(S_i) \qquad (3)$$

where Re(S_i) denotes the real component of the *i*th of *n* measurements of *S*.

3) Calculate the mean of the imaginary component, $\overline{\mathrm{Im}(S)}$, of the complex quantity, *S*, using:

$$\overline{\mathrm{Im}(S)} = \frac{1}{n}\sum_{i=1}^{n}\mathrm{Im}(S_i) \qquad (4)$$

where Im(S_i) denotes the imaginary component of the *i*th of *n* measurements of *S*.

4) Calculate the standard deviation in the mean of the real component, $s(\overline{\mathrm{Re}(S)})$, of the complex quantity, *S*, using:

$$s(\overline{\mathrm{Re}(S)}) = \sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}(\mathrm{Re}(S_i) - \overline{\mathrm{Re}(S)})^2} \qquad (5)$$

5) Calculate the standard deviation in the mean of the imaginary component, $s(\overline{\mathrm{Im}(S)})$, of the complex quantity, *S*, using:

$$s(\overline{\mathrm{Im}(S)}) = \sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}(\mathrm{Im}(S_i) - \overline{\mathrm{Im}(S)})^2} \qquad (6)$$

6) NO NOT use equations (1) and (2) to summarise complex data. Instead, if required, transform the calculated values into a polar representation (magnitude and phase) only after all computations have been performed on the data.

4. Correlation and the covariance matrix

Having shown that it is better to analyse complex-valued measurement data in terms of the real and imaginary components, consideration needs to be given to whether there is any correlation between the components. Correlation can be described as the degree to which variations in the two components are interrelated. For example, if a variation in some physical process (such as the misalignment of a microwave connection) causes both real and imaginary components of a given *S*-parameter to increase, then these components are said to be positively correlated. If, on the other hand, variation in the physical process causes one component to increase while the other component decreases, then this is termed negative correlation.

The degree of correlation between two measured mean components, $\bar{x}$ and $\bar{y}$, can be evaluated using the correlation coefficient, $r(\bar{x},\bar{y})$, which is calculated as follows:

$$r(\bar{x},\bar{y}) = \frac{s(\bar{x},\bar{y})}{s(\bar{x})\,s(\bar{y})} \tag{7}$$

where $s(\bar{x})$ and $s(\bar{y})$ are the standard deviation in the mean of *x* and *y*, respectively, and are given by:

$$s(\bar{x}) = \sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}(x_i-\bar{x})^2} \tag{8}$$

$$s(\bar{y}) = \sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}(y_i-\bar{y})^2} \tag{9}$$

and $s(\bar{x},\bar{y})$ is the covariance in the means of *x* and *y*, and is given by:

$$s(\bar{x},\bar{y}) = \frac{1}{n(n-1)}\sum_{i=1}^{n}(x_i-\bar{x})^2(y_i-\bar{y})^2 \tag{10}$$

An alternative way of expressing this information is as the covariance matrix in the mean:

$$\begin{pmatrix} s^2(\bar{x}) & s(\bar{x},\bar{y}) \\ s(\bar{y},\bar{x}) & s^2(\bar{y}) \end{pmatrix} \tag{11}$$

where $s(\bar{x},\bar{y}) = s(\bar{y},\bar{x})$, showing that the covariance matrix is symmetric.

Now, substituting Re(*S*) and Im(*S*) for *x* and *y*, respectively, we can therefore express the uncertainty in a complex measurand in two ways, either: (i) using the three terms; uncertainty in the real component, uncertainty in the imaginary component, and correlation coefficient, or (ii) as the covariance matrix.

A simple introduction to correlation and the use of the correlation coefficient in metrology has been presented in [5, 6].

EXAMPLE

The following example shows how we would evaluate and express a result for a complex measurand, along with its uncertainty, using the techniques presented in the previous sections. For simplicity, this example assumes that only random errors affect the

measurement process (*i.e.* no systematic errors), and these can be evaluated using statistical techniques.
Six repeat S_{11} (reflection coefficient) measurements[3] were made on a one-port device at an arbitrary frequency. These measurements are shown in the Table 1 below[4].

Measurement number	Real part of S_{11} (linear units)	Imaginary part of S_{11} (linear units)
1	0.1847	0.1866
2	0.1852	0.1924
3	0.2072	0.1925
4	0.2003	0.1880
5	0.2031	0.2080
6	0.2044	0.2233

Table 1: Six repeat S_{11} measurements of a one-port device at an arbitrary frequency.

Now, we use equation (3) to calculate the mean of the real part of S_{11}, equation (4) to calculate the mean of the imaginary part of S_{11}, equation (5) to calculate the standard deviation in the mean of the real part of S_{11}, equation (6) to calculate the standard deviation in the mean of the imaginary part of S_{11}, and equation (7) to calculate the correlation coefficient between means of the real and imaginary parts of S_{11}. These summary statistics are shown in Table 2.

Summary statistic	Real part of S_{11} (linear units)	Imaginary part of S_{11} (linear units)
Mean	0.1975	0.1985
Standard deviation in the mean	0.0041	0.0059
Correlation coefficient	+0.511	

Table 2: Summary statistics obtained from the measurements presented in Table 1.

Alternatively, we can present the results in terms of the means in the real and imaginary parts of S_{11}, along with the elements of the covariance matrix, as follows:

$S_{11} = 0.1975 + j\,0.1985$, with covariance matrix $\begin{pmatrix} 1.6541\times10^{-5}, & 1.2178\times10^{-5} \\ 1.2178\times10^{-5}, & 3.4347\times10^{-5} \end{pmatrix}$.

5. Geometric representation of uncertainty

Another consequence of considering the uncertainty in a complex-valued quantity, is the methods used to depict the uncertainty. In the case of a scalar measurand, the uncertainty can be thought of as a one-dimensional interval extending on either side of the mean measured value. However, for a complex-valued (vector) measurand, we need to think in terms of a two-dimensional *region* of uncertainty in the complex plane. Since the covariance matrix is positive definite, then an ellipse defines such a region: the size, aspect ratio and orientation with respect to the coordinate axes for the ellipse are governed by the elements of the covariance matrix (or the uncertainty in the real and imaginary components, along with the

[3] These 'measurements' are actually simulated measurements generated using a bivariate (*i.e.* complex-valued) normal data simulator, as described in [7].
[4] These values have been rounded to four decimal places, for clarity of presentation. Therefore subsequent summary values (Table 2) may contain an element of error due to rounding.

correlation coefficient). This is based on the assumption that the data emanates from a bivariate normal distribution.

To illustrate this, Figure 1 shows the uncertainty ellipse obtained from the example considered previously. This figure shows the six repeat measurements, as crosses in the plane, along with a square showing the mean value, and the uncertainty ellipse centred on the mean value. The ellipse has been scaled up to give a 95% confidence region.

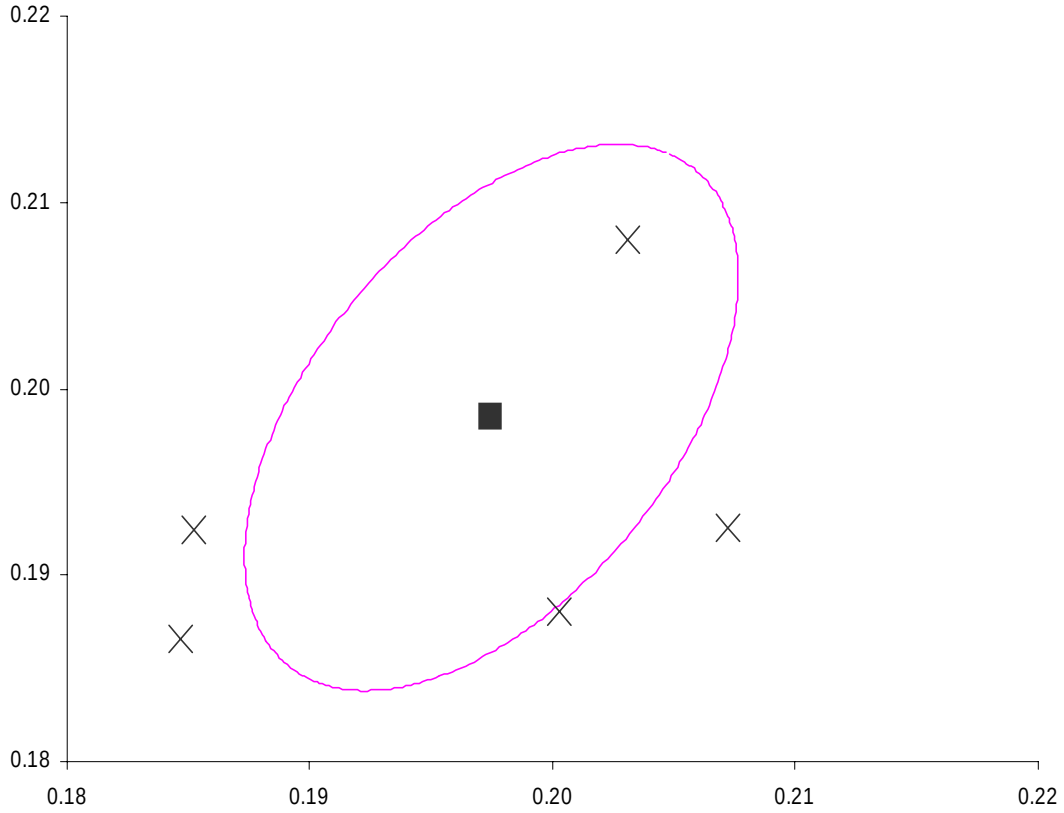

Figure 1: Elliptical 95% confidence region for example repeat measurements. Measured values are shown as crosses and the mean value is shown as a square.

A geometric representation of a complex-valued uncertainty can be very useful in observing and understanding trends in *S*-parameter uncertainties. For example, changes in the size, shape and orientation of the uncertainty ellipse can be examined as a function of the measurement frequency. Such an investigation has been presented in [8].

6. Reliability of a sample correlation coefficient

Having introduced the correlation coefficient as a measure of the interdependence between the real and imaginary components of a complex measurand, it is worth considering the reliability of a calculated correlation coefficient based on a given number of repeat measurements regarded as a sample from the population of all possible values. We would expect that, as the number of repeat measurements is increased, the sample correlation coefficient would become increasingly reliable, converging to the correlation coefficient characterising the underlying population of values.

Therefore, we wish to establish a confidence interval for the population correlation coefficient. A confidence interval provides a range of values which is likely to contain the 'true' underlying population correlation coefficient at a given level of confidence (for example, 95%). This is similar to a confidence interval assigned to a conventional measurement result. However, we cannot use the usual rules to establish a confidence interval based on normal distribution theory, because the distribution of the correlation coefficient is often very non-normal and asymmetric. This is because the correlation coefficient is a bounded quantity[5] having an upper bound of +1 and a lower bound of –1, *i.e.*

$$-1 \le r \le +1$$

and

$$-1 \le \rho \le +1$$

where r is the sample correlation coefficient and ρ the population correlation coefficient. For a detailed discussion on the distribution of the sample correlation coefficient, including a Monte Carlo study performed using a bivariate data simulator, see reference [7].

However, in 1921, Fisher [9] introduced a new statistic, z, which transforms any given distribution in r into a near-normal distribution for z[6]. Therefore, normal distribution theory can be applied to the z-statistic, obtaining confidence intervals, etc, which can then be transformed back to the equivalent confidence interval for the estimate of ρ.

The Fisher z-transformation is given by:

$$z = \frac{1}{2}\log_e \frac{1+r}{1-r} \tag{12}$$

and it follows that:

$$r = \tanh z \tag{13}$$

The standard deviation in z, $s(z)$, is approximately given by:

$$s(z) = \frac{1}{\sqrt{n-3}} \tag{14}$$

where n is the number of measurements used to calculate the correlation coefficient. Note that $s(z)$ is independent of the value of the population correlation coefficient. An expanded uncertainty, U, in z can be calculated as follows:

$$U = ks(z) \tag{15}$$

where k is a coverage factor chosen to provide the required level of confidence. For example, $k = 1.96$ provides a level of confidence of 95%.

Therefore, for a given calculated correlation coefficient, r, based on a series of n measurements, we can construct a confidence interval for the population correlation coefficient, ρ, as follows.

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[5] The boundaries (beyond which a value cannot exist) on the correlation coefficient scale cause a skewing to the distribution of values. There are similarities here with the boundary present of the *S*-parameter magnitude scale at zero, which also caused a skewing of the distribution of magnitude values (mentioned in section 2).

[6] The sampling distribution of z for a range of different values of ρ, and different samples sizes, has also been presented in [7].

1) Convert r to z using equation (12).
2) Calculate $s(z)$ using equation (14).
3) Calculate U using equation (15).
4) Use z and U to calculate the maximum and minimum values of z due to U, *i.e.* $z_{max} = z + U$, and $z_{min} = z - U$.
5) Convert z_{max} and z_{min} to the equivalent r_{max} and r_{min} using equation (13)
6) Use r, r_{max} and r_{min} to obtain a confidence interval for ρ, *i.e.* the upper interval is given by $r_{max} - r$, and the lower interval by $r - r_{min}$. It is necessary to calculate separately the upper and lower parts of the uncertainty interval because, in general, the overall uncertainty interval will be asymmetric.

EXAMPLE

In the earlier example, a correlation coefficient of +0.511 was calculated based on 6 repeat measurements. To find out the 95% confidence interval for the population correlation coefficient, ρ, we follow the six steps given above.

1) $$z = \frac{1}{2}\log_e \frac{1+r}{1-r} = \frac{1}{2}\log_e \frac{1+0.511}{1-0.511} = +0.564$$

2) $$s(z) = \frac{1}{\sqrt{n-3}} = \frac{1}{\sqrt{6-3}} = 0.577$$

3) $$U = ks(z) = 1.96 \times 0.577 = 1.131$$

4) $z_{max} = z + U = 0.564 + 1.131 = +1.695$, and
$z_{min} = z - U = 0.564 - 1.131 = -0.567$

5) $r_{max} = \tanh z_{max} = \tanh (+1.695) = +0.935$, and
$r_{min} = \tanh z_{min} = \tanh (-0.567) = -0.513$

6) The upper interval for r is given by $r_+ = r_{max} - r = +0.935 - 0.511 = 0.424$
the lower interval for r is given by $r_- = r - r_{min} = +0.511 - (-0.513) = 1.024$

Therefore

$$\text{Correlation coefficient, } \rho = +0.511 \left\{ \begin{matrix} +0.424 \\ \pm \\ -1.024 \end{matrix} \right\}$$

where the uncertainty provides an interval expected to contain the 'true' underlying population correlation coefficient at a 95% level of confidence.

Clearly, in the above example, the uncertainty interval for the correlation coefficient is asymmetric (as had been predicted previously). It is also worth noting that the uncertainty interval (*i.e.* 0.424 – (-1.024) = 1.448) is very large (bearing in mind that the total scale for a correlation coefficient extends from –1 to +1, *i.e.* a range of 2). Examining equations (14) and (15), we see that the uncertainty in z (and hence the uncertainty in ρ), for a given level of confidence, is related exclusively to the number of repeat measurements, n. Therefore, to

obtain a more reliable estimate of the correlation coefficient, more repeat measurements are needed.

However, on this occasion[7], we can test whether this interval actually does contain the 'true' value for the underlying population correlation coefficient. In fact, the 'true' value for the simulated data was set to be +0.8, which *is* contained within the calculated uncertainty interval. This gives confidence in this method of evaluating an uncertainty interval for the correlation coefficient.

Finally, having established an indication of the reliability of the estimated correlation coefficient (based on a confidence interval), we see that the degree of precision implied by expressing the correlation coefficient to three decimal places is misleading. A more pragmatic representation for such an unreliable estimate would be to present the result to just one decimal place, *i.e.*

$$\rho = +0.5 \left\{ \begin{matrix} \pm 0.4 \\ 1.0 \end{matrix} \right\}$$

7. Extension to multiport devices

The above approach to evaluating and expressing the uncertainty in a complex *S*-parameter measurement can be extended to deal with multiple *S*-parameters measurements made simultaneously on multiport, or *n*-port, devices. For each real and imaginary component of each *S*-parameter, the mean and standard deviation in the mean is calculated, along with the correlation coefficient for each real and imaginary pair. However, in addition to this, correlation *between* the *S*-parameters (both the real and imaginary components) also needs to be considered.

For example, for a two-port device, four complex *S*-parameters are measured. This results in an (8 × 8) covariance matrix, containing 28 off-diagonal elements, resulting in 28 correlation coefficients representing all possible combinations of pairs of *S*-parameter components. Some of the elements of this covariance matrix are shown below (assuming the *S*-parameters are arranged as S_{11}, S_{21}, S_{12} and S_{22}), using dots to indicate the position of the missing elements.

$$\begin{pmatrix} u^2(\mathrm{Re}(S_{11})) & u(\mathrm{Re}(S_{11}),\mathrm{Im}(S_{11})) & . & . & . & . & u(\mathrm{Re}(S_{11}),\mathrm{Im}(S_{22})) \\ u(\mathrm{Im}(S_{11}),\mathrm{Re}(S_{11})) & u^2(\mathrm{Im}(S_{11})) & . & . & . & . & u(\mathrm{Im}(S_{11}),\mathrm{Im}(S_{22})) \\ . & . & u^2(\mathrm{Re}(S_{21})) & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ u(\mathrm{Re}(S_{22}),\mathrm{Re}(S_{11})) & . & . & . & . & . & u(\mathrm{Re}(S_{22}),\mathrm{Im}(S_{22})) \\ u(\mathrm{Im}(S_{22}),\mathrm{Re}(S_{11})) & . & . & . & . & . & u^2(\mathrm{Im}(S_{22})) \end{pmatrix}$$

[7] Remembering that the original measurement data was generated using a bivariate normal data simulator, which enables the population correlation coefficient to be specified, and hence this becomes our 'true' value.

In general, to fully characterise an n-port device, there will be n^2 S-parameter measurements, with a $(2n^2 \times 2n^2)$ covariance matrix, resulting in $n^2(2n^2 - 1)$ correlation coefficients. Table 3 gives values for these characteristics for complex one-, two-, three- and four-port measurements. In terms of providing a geometric representation of the uncertainty, this would require displaying the uncertainty ellipsoid in $2n^2$-dimensional space!

Number of ports	Number of complex S-parameters	Size of covariance matrix	Number of correlation coefficients
1	1	(2 × 2)	1
2	4	(8 × 8)	28
3	9	(18 × 18)	153
4	16	(32 × 32)	496

Table 3: showing the number of values required to characterise n-port devices (n = 1, 2, 3, 4).

8. Coverage factors

Having established the standard uncertainty for all the components in a measurand, it is conventional to apply a multiplying factor (often called a *coverage factor*) to expand the uncertainty to correspond to a given confidence probability. For example, for scalar quantities, it is conventional to use a coverage factor of $k = 2$, to give a 95% confidence interval[8].

However, for multiple complex-valued measurands, a coverage factor of $k = 2$ no longer yields a confidence probability of 95%. For example, a value of approximately $k = 2.45$ is required to provide a 95% level of confidence for a single complex S-parameter (such as a reflection coefficient). This k-factor is obtained from a table of values for a χ^2-distribution with two degrees of freedom [10], since a degree of freedom is required for each component of the complex measurand. We can therefore derive further k-factors for n-port measurements in a similar way. These are given in Table 4, for complex one-, two-, three- and four-port measurements.

Number of ports	k-factor required to give a 95% level of confidence
1	2.45
2	3.94
3	5.37
4	6.80

Table 4: showing coverage factors required to produce 95% confidence intervals for complex multi-port S-parameter measurements.

9. Reporting complex-valued results

One of the traditional methods used to report the results of a measurement is to present the results on a calibration certificate. This is the method used currently by calibration laboratories providing measurements traceable to national measurement standards. However, the previous sections of this paper have shown that the amount of information required to

[8] Assuming that the underlying distribution characterising the measurand is normal and that the uncertainty is known with a sufficiently high number of degrees of freedom.

fully characterise complex *S*-parameter measurements can be very large. In the case of a single complex *S*-parameter, the traditional method of reporting may still be applicable. For example, using a table listing the required parameters for each measurement frequency. This is shown in Table 5, presenting the results obtained for the example given earlier in this paper[9].

Frequency	Real component	Imaginary component	Uncertainty in real component	Uncertainty in imaginary component	Correlation coefficient
f_i	0.1978	0.1985	0.0100	0.0145	+0.5

Table 5: showing the example results as they might appear on a certificate of calibration.

The above result could well be one of many made at different measurement frequencies (*i.e.* the *i*th result of *n*, where *n* could be of the order of several hundreds). This leads to a calibration certificate consisting of many pages of numbers. Such a large amount of information may well overwhelm the end-user of the calibration results. This is particularly likely when a multi-port device is measured, when the amount of data even at a single frequency does not lend itself to easy presentation in the traditional calibration certificate format.

These concerns have led to research into using data visualisation techniques [11] to present results from *S*-parameter measurements. In particular, the use of computer animation has enabled geometric representations of uncertainty (*i.e.* ellipses) to be depicted sequentially, running in real time. At minimum, such techniques can be very useful in studying trends in multiple complex *S*-parameters, and may lead to new methods of presenting and reporting such results.

10. Using complex-valued results

Finally, we consider methods of using the results of multiple complex *S*-parameter measurements where the characteristics have been presented following the methods given in this paper. It is assumed that a recipient of such measurement results is required subsequently to use the results to evaluate, or characterise, their own measurement system (or systems). In effect, the *S*-parameter measurements become input quantities to the measurement process, resulting in an output quantity (or quantities).

Generalised measurement system models, including complex-valued models, have been presented in [12]. Such models can be applied to propagate the uncertainties in the input quantities to those in the output quantities. If we represent each of *n* input *S*-parameters by z_i ($i = 1, \ldots, n$), and *m* (assumed here to be complex) output quantities, by ξ_j ($j = 1, \ldots, m$) then we have:

$$\boldsymbol{\xi} = f(\mathbf{z}) \tag{16}$$

The uncertainty in $\boldsymbol{\xi}$ is represented by its ($2m \times 2m$) covariance matrix, V_ξ,

$$V_\xi = J_z V_z J_z^T \tag{17}$$

[9] Note that the standard uncertainties evaluated previously have been converted into expanded uncertainties at the 95% level of confidence (using $k = 2.45$), as this level of confidence is often used by calibration laboratories.

where V_z is the ($2n \times 2n$) covariance matrix for the input *S*-parameters, and J_z is the ($2m \times 2n$) Jacobian matrix containing the partial derivatives of f with respect to $\boldsymbol{z}$. All uncertainty components and correlation coefficients are derived from the elements of V_ξ.

This approach can be applied to all situations where the results of complex *S*-parameters are used to establish some other measurement parameter, *e.g.* the effects of complex mismatch on power, attenuation and noise measurements. The approach has also been applied to the propagation of uncertainties in the calibration and measurements of a microwave reflectometer [13].

10. Conclusions

This paper has presented methods for evaluating and expressing the uncertainty in complex *S*-parameter measurements. The methods represent extensions to the current recommended practices [1 - 3], making them suitable for complex-valued quantities. The methods have been applied to single *S*-parameters (such as reflection coefficients) as well as multiple *S*-parameters used to characterise *n*-port devices. It has been recommended that *S*-parameters should be expressed, in general, in terms of real and imaginary components, converting to a magnitude and phase format, if required, only after all numerical manipulations have been performed on the data.

A generalised treatment of the uncertainty in complex measurands has included the use of correlation coefficients and covariance matrices to fully describe the uncertainty, along with an indication of the reliability of a sample correlation coefficient. These concepts have been extended to *n*-port *S*-parameter measurements. Information has also been given on choosing coverage factors to achieve a given level of confidence for multi-port complex measurements.

Finally, consideration has been given to methods of expressing and using complex-valued results. It has been shown that it may be more appropriate in future to abandon the conventional method of presenting *S*-parameter results on calibration certificates and instead implement electronic techniques (such as data visualisation) using computers.

11. Acknowledgement

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12. References

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