Over-determined calibration schemes for RF network analysers employing generalised distance regression

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Abstract

This paper presents over-determined calibration schemes for vector network analysers (i.e. schemes involving more than the traditional three calibration standards). Generalised Distance Regression is used to obtain estimates for the three complex-valued calibration coefficients of the network analyser from the indicated and 'true'[1] complex-valued reflection coefficients of the standards. Account is taken of the uncertainties associated with the indicated and 'true' reflection coefficients of the standards to establish the uncertainty in the calibration coefficients and subsequently the uncertainty in the reflection coefficient measurements made using the calibrated network analyser. Typical results are presented to demonstrate the performance of these new calibration schemes compared with more traditional methods.

1 Introduction

In recent years, much work has been done to improve the performance of the methods used to calibrate Vector Network Analysers (VNAs). This has included using Least Squares approaches to deal with one-port calibrations [1] and Generalised Regression techniques for two-port calibrations [2]. The work presented in this paper is a further contribution to this subject and applies Generalised Regression techniques to the one-port calibration problem, specifically at RF (a term used here to indicate frequencies in the MHz region). This frequency range is of great interest commercially and also coincides with the region where calibration schemes employing transmission lines as standards (e.g. TRL [3] and LRL [4]) become ineffective. There is therefore a significant need to address the VNA calibration problem in this frequency region.

Recent work at the UK's National Physical Laboratory (NPL) has concentrated on producing impedance 'standards' at RF by fitting polynomials to measured data, with uncertainties, at DC (i.e. resistance) and microwave frequencies (i.e. complex reflection coefficient)[2]. The polynomials are then used to interpolate the characteristics, including the uncertainties, of the standards at any intermediate frequency (i.e. the RF region). This work has been discussed in detail in [7, 8]. In

[1] The term 'true' is used in this paper to imply that a value is assumed known (albeit with an element of uncertainty).

[2] The measurements at microwave frequencies are made using NPL's Primary Impedance Microwave Measurement System (PIMMS) [5, 6]. This system uses transmission lines (such as coaxial air lines) as the standards for impedance at microwave frequencies realised using the TRL and LRL calibration techniques.

general, this approach makes no assumptions about the physical nature of the standards being characterised and has therefore been extended to include ‘any’ device (i.e. devices other than the traditional Short-Open-Load standards used for one-port calibrations) [9]. This therefore has made available more than the minimum number of standards (i.e. three) needed to effect the VNA calibration.

The work presented in this paper describes the use of regression techniques to make full use of additional information made available by the use of more than the minimum number of standards, and the associated uncertainty, to calibrate the VNA. To do this, techniques employing Generalised Distance Regression [10] have been implemented for one-port VNA calibrations at RF. However, it is recognised that these techniques are capable of being extended to two-port measurements and to other frequency bands and other transmission media (including on-wafer measurements at microwave and millimetre-wave frequencies).

This paper begins by formulating the VNA calibration in terms of a regression problem. This is followed by descriptions of such problems in terms of a hierarchy of approaches utilising regression and a comparison with Weighted Least Squares problems. Subsequent sections describe: the techniques used for characterising the standards; evaluating and expressing the uncertainty of measurement; choosing between a range of candidate calibration schemes; and over-determining the VNA calibration. Some typical results demonstrate the advantages of applying these methods over the more conventional VNA calibration schemes.

2 VNA calibration as a regression problem

Suppose a device is connected to the test port of a one-port VNA. The relationship between the ‘true’ value of the voltage reflection coefficient (VRC) of the device, Γ, and the value indicated by the VNA, *w*, is [11]

$$w = \frac{a\Gamma + b}{c\Gamma + 1} \qquad (1)$$

where *a*, *b* and *c* are the calibration coefficients of the VNA. In equation (1) all the quantities are complex-valued and so we write

$$w = u + jv, \quad \Gamma = x + jy, \quad a = a_R + ja_I, \quad b = b_R + jb_I, \quad c = c_R + jc_I .$$

In order to calibrate the VNA (i.e. to obtain estimates of the calibration coefficients *a*, *b* and *c*), a number of calibration standards with known ‘true’ VRCs are connected to the VNA test port and the corresponding indicated VRCs are observed. Since there are three unknown parameters in the model (1), values can be obtained for *a*, *b* and *c* using three calibration standards. However, in order to reduce the influence of the uncertainties in the ‘true’ and indicated VRCs, it is advantageous to calibrate the VNA with more than three standards i.e. to use an over-determined calibration scheme.

Suppose m calibration standards are to be used ($m \geq 3$). Let the 'true' VRC for the i^{th} standard (i = 1, …, m) be $\Gamma_i = x_i + j\,y_i$ and the corresponding indicated VRC be $w_i = u_i + j\,v_i$. Assemble the data for the i^{th} standard (i.e. the real and the imaginary components of w_i and Γ_i) into the vector $\mathbf{S_i}$

$$\mathbf{S_i} = \begin{pmatrix} u_i & v_i & x_i & y_i \end{pmatrix}^{\mathrm{T}}$$

where $^{\mathrm{T}}$ denotes vector/matrix transpose. The data vector for the entire calibration is obtained by concatenating the data vectors for each of the calibration standards into a single $4m$ dimensional vector

$$\mathbf{S} = \begin{pmatrix} \mathbf{S_1}^{\mathrm{T}} & \cdots & \mathbf{S_m}^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} u_1 & v_1 & x_1 & y_1 & \cdots & u_m & v_m & x_m & y_m \end{pmatrix}^{\mathrm{T}}.$$

The uncertainty in the data vector $\mathbf{S}$ is expressed by means of a $4m \times 4m$ covariance matrix V,

$$V = \begin{bmatrix} V_{11} & V_{12} & \cdots & V_{1m} \\ V_{21} & V_{22} & & \\ \vdots & & \ddots & \vdots \\ V_{m1} & & \cdots & V_{mm} \end{bmatrix}$$

where the V_{ij} are 4×4 sub-matrices containing variances and covariances. For example, the sub-matrix V_{11} that is located on the main diagonal of V is the covariance matrix for calibration standard 1 (i.e. for the vector $\mathbf{S_1}$)

$$V_{11} = \begin{bmatrix} Var(u_1) & Cov(u_1, v_1) & Cov(u_1, x_1) & Cov(u_1, y_1) \\ Cov(v_1, u_1) & Var(v_1) & Cov(v_1, x_1) & Cov(v_1, y_1) \\ Cov(x_1, u_1) & Cov(x_1, v_1) & Var(x_1) & Cov(x_1, y_1) \\ Cov(y_1, u_1) & Cov(y_1, v_1) & Cov(y_1, x_1) & Var(y_1) \end{bmatrix}.$$

Similarly, the sub-matrix V_{12} that is located in an off-diagonal position in V contains covariances between calibration standards 1 and 2 (i.e. between the elements of vectors $\mathbf{S_1}$ and $\mathbf{S_2}$)

$$V_{12} = \begin{bmatrix} Cov(u_1, u_2) & Cov(u_1, v_2) & Cov(u_1, x_2) & Cov(u_1, y_2) \\ Cov(v_1, u_2) & Cov(v_1, v_2) & Cov(v_1, x_2) & Cov(v_1, y_2) \\ Cov(x_1, u_2) & Cov(x_1, v_2) & Cov(x_1, x_2) & Cov(x_1, y_2) \\ Cov(y_1, u_2) & Cov(y_1, v_2) & Cov(y_1, x_2) & Cov(y_1, y_2) \end{bmatrix}.$$

The matrix V is symmetric, and so the sub-matrices on the main diagonal of V are symmetric

$$V_{ii}^{\mathrm{T}} = V_{ii}$$

and the sub-matrices in off-diagonal positions in V satisfy

$$V_{ij}^{\mathrm{T}} = V_{ji} .$$

Given the data vector **S** and the corresponding covariance matrix V, the calibration problem is to determine best estimates for the values of the real and imaginary components of the calibration coefficients

$$\mathbf{d} = (a_R, a_I, b_R, b_I, c_R, c_I)^{\mathrm{T}}$$

and an associated 6 × 6 covariance matrix $V(\mathbf{d})$.

Because w_i and Γ_i ($i = 1, \ldots, m$) include an element of uncertainty they will not satisfy equation (1) exactly. Let the corresponding "model values" that do satisfy equation (1) be $w^*{}_i = u^*{}_i + jv^*{}_i$ and $\Gamma^*{}_i = x^*{}_i + j\, y^*{}_i$. The $4m$ dimensional vector of model values is

$$\mathbf{S}^* = \left(u_1^* \quad v_1^* \quad x_1^* \quad y_1^* \quad \cdots \quad u_m^* \quad v_m^* \quad x_m^* \quad y_m^*\right)^{\mathrm{T}} .$$

Define **e** to be the vector comprising the residual errors between the data vector and the corresponding vector of model values

$$\begin{aligned}\mathbf{e} &= \mathbf{S} - \mathbf{S}^* \\ &= \left(u_1 - u_1^* \quad v_1 - v_1^* \quad x_1 - x_1^* \quad y_1 - y_1^* \quad \cdots \quad u_m - u_m^* \quad v_m - v_m^* \quad x_m - x_m^* \quad y_m - y_m^*\right)^{\mathrm{T}} .\end{aligned}$$

Estimates of a, b and c can then be obtained by solving the following constrained minimisation problem

$$\min \mathbf{e}^{\mathrm{T}} V^{-1} \mathbf{e} \quad \text{……………………..…………………(2)}$$

with respect to the $4m + 6$ variables

$$\left\{u_1^*, v_1^*, x_1^*, y_1^*, \cdots, u_m^*, v_m^*, x_m^*, y_m^*, a_R, a_I, b_R, b_I, c_R, c_I\right\}$$

subject to the constraints

$$w_i^* = \frac{a\Gamma_i^* + b}{c\Gamma_i^* + 1}, \; i = 1, \ldots, m.$$

The constraints express u_i* and v_i* in terms of the other variables. They are used to eliminate the variables u_i* and v_i* in the definition of the residual vector, giving the unconstrained minimisation problem

$$\min \mathbf{e}^{\mathrm{T}} V^{-1} \mathbf{e} \quad \text{…………………………………(3)}$$

with respect to the $2m + 6$ variables

$$\mathbf{p} = \left(x_1^*, y_1^*, \cdots, x_m^*, y_m^*, a_R, a_I, b_R, b_I, c_R, c_I \right)^{\mathrm{T}}$$

Finally, by means of the Cholesky decomposition [12] of the covariance matrix V, the problem can be reduced to the following *standard unconstrained non-linear least squares problem*

$$\min \tilde{\mathbf{e}}^{\mathrm{T}} \tilde{\mathbf{e}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(4)$$

with respect to the $2m + 6$ components of the vector $\mathbf{p}$ where

$$\tilde{\mathbf{e}} = L^{-1}\, \mathbf{e}$$

and L is the (lower triangular) Cholesky factor of the covariance matrix V

$$V = L\, L^{\mathrm{T}} .$$

The problem is non-linear because the components of $\tilde{\mathbf{e}}$ are non-linear functions of the components of $\mathbf{p}$. The non-linear least squares problem (4) can be solved, for example, using a Gauss-Newton algorithm [13][3].

3 Hierarchy of regression problems – structure of the covariance matrix

In VNA calibration, the errors in distinct standards are generally assumed to be uncorrelated i.e. the errors in data vectors $\mathbf{S_i}$ and $\mathbf{S_j}$ $(i \neq j)$ are assumed to be uncorrelated. As a result of this, the sub-matrices V_{ij} and V_{ji} in the covariance matrix V are zero and so the matrix V is block diagonal. Under these circumstances, the VNA calibration problem (3) is an example of a Generalised Distance Regression (GDR) problem [10] and the structure of the covariance matrix V is as follows.

$$V = \begin{bmatrix} \underbrace{\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}}_{V_{11}} & & & & \mathbf{0} \\ & \underbrace{\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}}_{V_{22}} & & & \\ & & \ddots & & \\ & & & \underbrace{\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}}_{V_{m-1,\, m-1}} & \\ \mathbf{0} & & & & \underbrace{\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}}_{V_{m,\, m}} \end{bmatrix}$$

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[3] Reference [13] considers the model $\Gamma = \dfrac{aw+b}{cw+1}$. This is mathematically equivalent to that in equation (1) except that the calibration coefficients a, b and c take different values in the two models.

It is also usually assumed that for standard *i*, the errors in the 'true' VRC Γ_i are uncorrelated with the errors in the indicated VRC w_i and as a result the sub-matrix V_{ii} in the matrix V is itself block-diagonal as represented below where $V(w_i)$ and $V(\Gamma_i)$ are the covariance matrices for w_i and Γ_i:

$$V_{ii} = \begin{bmatrix} \underbrace{\begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix}}_{V(w_i)} & 0 \\ 0 & \underbrace{\begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix}}_{V(\Gamma_i)} \end{bmatrix}$$

More generally, if full correlation is assumed between the errors in standards *i* and *j* (i.e. between data vectors $\mathbf{S_i}$ and $\mathbf{S_j}$), the sub-matrices V_{ij} and V_{ji} (as well as the sub-matrices V_{ii} and V_{jj}) are full matrices and hence the covariance matrix V is a full matrix and the calibration problem (3) is an example of a Generalised Gauss-Markov Regression problem [10]. Although this degree of generality has not been considered to date, it might be argued, for example, that correlation between the errors in the indicated VRCs for different standards should be considered (and similarly for the 'true' VRCs for different standards) since the values for different standards derive from the same instrument on the same occasion.

If the errors in the components of the data vector $\mathbf{S}$ are assumed to be completely uncorrelated, the covariance matrix V is diagonal and the calibration problem (3) is an example of an Orthogonal Distance Regression problem [10]. The structure of the covariance matrix is then as follows.

$$V = \begin{bmatrix} \times & & & & \\ & \times & & 0 & \\ & & \ddots & & \\ & 0 & & \times & \\ & & & & \times \end{bmatrix}$$

This class of problem is probably too restricted to be of general use in VNA calibration as it is not possible to allow for any correlation between the real and imaginary components of an indicated VRC w (or of a 'true' VRC Γ).

4 Comparison with Weighted Least Squares

In order to compare the above Generalised Distance Regression scheme for VNA calibration with a Weighted Least Squares calibration scheme [1], it is convenient to consider a different ordering of the data vector as follows. Let $\mathbf{S}_\Gamma$ be the vector of the real and imaginary components of the 'true' VRCs of the calibration standards

$$\mathbf{S}_\Gamma = (x_1 \quad y_1 \quad \cdots \quad x_m \quad y_m)^{\mathrm{T}}$$

and let $\mathbf{S_W}$ be the vector of the real and imaginary components of the corresponding indicated VRCs

$$\mathbf{S_W} = \begin{pmatrix} u_1 & v_1 & \cdots & u_m & v_m \end{pmatrix}^{\mathrm{T}} .$$

Let **S'** denote the data vector for the entire calibration obtained by concatenating the vectors $\mathbf{S_W}$ and $\mathbf{S_\Gamma}$ into a single vector

$$\mathbf{S}' = \begin{pmatrix} \mathbf{S_W}^{\mathrm{T}} & \mathbf{S_\Gamma}^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} u_1 & v_1 & \cdots & u_m & v_m & x_1 & y_1 & \cdots & x_m & y_m \end{pmatrix}^{\mathrm{T}} .$$

Note that the data vector **S'** contains the same information as the data vector **S** but in a different order. The uncertainty in the data vector **S'** is expressed by means of a $4m \times 4m$ covariance matrix, *V'*

$$V' = \begin{bmatrix} V_W & V_{W\Gamma} \\ V_{\Gamma W} & V_\Gamma \end{bmatrix}$$

where V_W, V_Γ, $V_{W\Gamma}$ and $V_{\Gamma W}$ are $2m \times 2m$ sub-matrices containing variances and covariances. The sub-matrices V_W and V_Γ which appear on the main diagonal of *V'* are the covariance matrices for vectors $\mathbf{S_W}$ and $\mathbf{S_\Gamma}$ respectively. The sub-matrices $V_{W\Gamma}$ and $V_{\Gamma W}$ which appear off the main diagonal of *V'* contain covariances between the components of vectors $\mathbf{S_W}$ and $\mathbf{S_\Gamma}$[4].

In Weighted Least-Squares and in (ordinary) Gauss-Markov Regression, it is assumed that the stimulus variables (corresponding here to the Γ values) are known exactly and are subject to no uncertainty and that they are uncorrelated with the response variables (corresponding to the *w* values). It follows that the sub-matrices V_Γ, $V_{W\Gamma}$ and $V_{\Gamma W}$ are all zero, and

$$V' = \begin{bmatrix} V_W & 0 \\ 0 & 0 \end{bmatrix}.$$

In Weighted Least Squares, the response variables (*w* values) are assumed to be uncorrelated corresponding to a diagonal covariance matrix V_W whereas, in ordinary Gauss-Markov Regression, correlation between the response variables (*w* values) is taken into account resulting in a full covariance matrix V_W [10].

In [1] both ordinary Gauss-Markov Regression and Weighted Least Squares are considered. However, these are somewhat limited, compared to Generalised Distance Regression, in that the uncertainties in the Γ values are not taken into account since they are assumed to be zero. This will affect the overall evaluation of the uncertainties in the calibration process as well as the estimates of the calibration coefficients.

[4] The matrix *V'* can be obtained from the matrix *V* by a re-ordering of its rows and columns.

5 Characterisation of VNA calibration standards

Previous papers [7, 8] described an RF impedance measurement system based on a one-port VNA in which the short-circuit, open-circuit and matched load calibration standards are each characterised by a polynomial fitted to measured data. The polynomial fit is based on VRC measurements at multiple microwave frequencies and either a simple model of the DC behaviour of the device (for a short-circuit or an open-circuit) or a DC resistance measurement (for a matched load). As described in [9], this procedure can be generalised to apply to any arbitrary one-port device. For a more general device, a polynomial fit is obtained based on VRC measurements at microwave frequencies and a resistance measurement at DC.

The polynomial fits allow the complex-valued VRC of the arbitrary one-port devices and the corresponding uncertainties (as covariance matrices) to be determined at frequencies between DC and the maximum RF measurement frequency. The VRC measurements at microwave frequencies are carried out using a two-port VNA calibrated by means of a TRL calibration scheme [3]. The use of TRL is not practical at lower frequencies (i.e. RF) and so the polynomial fits are used here to obtain interpolated VRC values enabling the devices to be used subsequently as calibration standards at RF.

6 Uncertainty of measurement

The flow graph in Figure 1 illustrates the propagation of uncertainty through the model of a one-port VNA (equation (1)) both during the calibration of the VNA and also during the subsequent measurement of the VRC of a device under test (DUT). The quantities involved are vector-valued and so the uncertainties are expressed throughout by means of covariance matrices [14].

Fundamental to the uncertainty calculations described here is the law of propagation of uncertainty expressed in matrix form [14, 15]

$$V(\mathbf{y}) = JV(\mathbf{x})J^{\mathrm{T}} \qquad (5)$$

which relates the covariance matrix of the output vector **y** to the covariance matrix of the input vector **x** by means of the Jacobian matrix J of the transformation from **x** to **y**.

The uncertainty in the calibration coefficients of the VNA is obtained from the uncertainty in the calibration data vector **S** i.e. in the ‘true’ and indicated VRCs of all the calibration standards. The uncertainty in the corrected VRC of the DUT is obtained from the uncertainty in the calibration coefficients and in the indicated VRC of the DUT with its own associated 2 × 2 covariance matrix representing the random errors induced by the measurement process (e.g. due to connector repeatability, instrument noise, etc). Further details are given in Sections 6.1 and 6.2, below.

6.1 Uncertainty in the estimated calibration coefficients of the VNA

The 2 × 2 covariance matrix, $V(\Gamma_i)$, for the ‘true’ VRC of calibration standard i, Γ_i, is obtained from the polynomial fit to the measured VRC of standard i described in section 5. The 2×2 covariance matrix, $V(w_i)$, for the indicated VRC of calibration standard i, w_i, is obtained from the statistical analysis of a number of repeat measurements. Assuming there to be no correlation between Γ_i and w_i, the 4×4 covariance matrix of data vector $\mathbf{S_i}$, V_{ii}, is the matrix with $V(w_i)$ and $V(\Gamma_i)$ as sub-matrices on the main diagonal and with zeros elsewhere

$$V_{ii} = \begin{pmatrix} V(w_i) & 0 \\ 0 & V(\Gamma_i) \end{pmatrix}.$$

Given the covariance matrices for the m calibration standards V_{ii} ($i = 1$ to m), the covariance matrix, V, of the calibration data vector $\mathbf{S}$ is formed with the sub-matrices V_{ii} on the main diagonal, and the off-diagonal sub-matrices V_{ij} $(i \neq j)$ assumed to be zero

$$V = \begin{pmatrix} V_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & V_{mm} \end{pmatrix}.$$

Next the Cholesky factor L of the covariance matrix V is calculated. The covariance matrix, $V(\mathbf{p})$, of the vector $\mathbf{p}$ is calculated from [13]

$$V(\mathbf{p}) = \left(J^{\mathrm{T}} J\right)^{-1} \qquad (6)$$

where J is the $4m \times 2m+6$ Jacobian matrix containing the derivatives of the components of the vector $\tilde{\mathbf{e}}$ ($\tilde{\mathbf{e}} = L^{-1}\mathbf{e}$) with respect to the components of the vector $\mathbf{p}$ evaluated at the solution to the Generalised Distance Regression problem

$$J = \begin{bmatrix} \frac{\partial \tilde{e}_1}{\partial x_1^*} & \frac{\partial \tilde{e}_1}{\partial y_1^*} & \frac{\partial \tilde{e}_1}{\partial x_2^*} & \frac{\partial \tilde{e}_1}{\partial y_2^*} & \cdots & \frac{\partial \tilde{e}_1}{\partial c_R} & \frac{\partial \tilde{e}_1}{\partial c_I} \\ \vdots & & & & & & \vdots \\ \frac{\partial \tilde{e}_{4m}}{\partial x_1^*} & \frac{\partial \tilde{e}_{4m}}{\partial y_1^*} & \frac{\partial \tilde{e}_{4m}}{\partial x_2^*} & \frac{\partial \tilde{e}_{4m}}{\partial y_2^*} & \cdots & \frac{\partial \tilde{e}_{4m}}{\partial c_R} & \frac{\partial \tilde{e}_{4m}}{\partial c_I} \end{bmatrix}.$$

The 6×6 sub-matrix of $V(\mathbf{p})$ comprising its last 6 rows and 6 columns contains the covariance matrix $V(\mathbf{d})$ associated with the values of the calibration coefficients $\mathbf{d}$. The covariance matrix $V(\mathbf{d})$ therefore expresses the uncertainty in the estimated calibration coefficients for the VNA. Equation (6) applies because it is assumed that the model and the data and its associated covariance matrix are consistent. This is checked by model validation, i.e. calculating a chi-squared statistic and comparing with its expected value.

6.2 Uncertainty in the measured VRC of the DUT

Once the VNA has been calibrated, the VRC of a DUT can be measured. The corrected VRC of the DUT, Γ_0, is related to the indicated VRC of the DUT, w_0, as follows

$$\Gamma_0 = \frac{-w_0 + b}{cw_0 - a} \quad \text{..................................(7)}$$

which can be written

$$\Gamma_0 (cw_0 - a) + w_0 - b = 0 .$$

All the quantities in this equation are complex-valued. The calibration coefficients a, b and c are as defined previously and we can write

$$w_0 = w_R + jw_I , \quad \Gamma_0 = \Gamma_R + j\Gamma_I .$$

The real and imaginary parts of the above equation are

$$g_1(\mathbf{\Gamma_0}, \mathbf{q}) = \Gamma_R (c_R w_R - c_I w_I - a_R) - \Gamma_I (c_I w_R + c_R w_I - a_I) + (w_R - b_R) = 0$$

$$g_2(\mathbf{\Gamma_0}, \mathbf{q}) = \Gamma_R (c_I w_R + c_R w_I - a_I) - \Gamma_I (c_R w_R - c_I w_I - a_R) + (w_I - b_I) = 0$$

where g_1 and g_2 are the component functions of a vector-valued function $\mathbf{g}$ (where $\mathbf{g} = (g_1, g_2)^{\mathrm{T}} = \mathbf{0}$). Define the vectors $\mathbf{\Gamma_0}$ and $\boldsymbol{w_0}$ containing the real and imaginary components of Γ_0 and w_0 respectively

$$\mathbf{\Gamma_0} = (\Gamma_R \quad \Gamma_I)^{\mathrm{T}}$$

$$\mathbf{w_0} = (w_R \quad w_I)^{\mathrm{T}} .$$

Let the vector containing the real and imaginary components of w_0 and the calibration coefficients $\mathbf{d}$ be $\mathbf{q}$

$$\mathbf{q} = \left(\mathbf{w_0}^{\mathrm{T}} \quad \mathbf{d}^{\mathrm{T}}\right)^{\mathrm{T}} = (w_R \quad w_I \quad a_R \quad a_I \quad b_R \quad b_I \quad c_R \quad c_I)^{\mathrm{T}} .$$

The 8 × 8 covariance matrix of the vector $\mathbf{q}$ is given by

$$V(\mathbf{q}) = \begin{pmatrix} V(\mathbf{w_0}) & V_{wd} \\ V_{dw} & V(\mathbf{d}) \end{pmatrix}$$

where $V(\mathbf{w_0})$ is the 2 × 2 covariance matrix of the vector $\mathbf{w_0}$ estimated from repeat measurements and $V(\mathbf{d})$ is the 6 × 6 covariance matrix of the vector $\mathbf{d}$ of calibration coefficients obtained as described in section 6.1. The sub-matrices V_{wd} and V_{dw} contain covariances between the elements of the vector $\mathbf{w_0}$ and elements of the vector $\mathbf{d}$ and these are assumed to be zero.

The 2 × 2 covariance matrix of the vector $\mathbf{\Gamma_0}$ is given by [13]

$$V(\mathbf{\Gamma_0}) = CV(\mathbf{q})C^{\mathrm{T}} \quad \text{.............................(8)}$$

where C is the 2 × 8 matrix that solves the linear equations

$$\frac{\partial \mathbf{g}}{\partial \mathbf{\Gamma_0}} C = \frac{\partial \mathbf{g}}{\partial \mathbf{q}}$$

where the Jacobian matrices $\partial\mathbf{g}/\partial\mathbf{\Gamma_0}$ and $\partial\mathbf{g}/\partial\mathbf{q}$ contain partial derivatives of the components of the vector-valued function $\mathbf{g}$ with respect to the components of the vectors $\mathbf{\Gamma_0}$ and $\mathbf{q}$ respectively

$$\frac{\partial \mathbf{g}}{\partial \mathbf{\Gamma_0}} = \begin{pmatrix} \frac{\partial g_1}{\partial \Gamma_R} & \frac{\partial g_1}{\partial \Gamma_I} \\ \frac{\partial g_2}{\partial \Gamma_R} & \frac{\partial g_2}{\partial \Gamma_I} \end{pmatrix}$$

$$\frac{\partial \mathbf{g}}{\partial \mathbf{q}} = \begin{pmatrix} \frac{\partial g_1}{\partial w_R} & \frac{\partial g_1}{\partial w_I} & \frac{\partial g_1}{\partial a_R} & \frac{\partial g_1}{\partial a_I} & \frac{\partial g_1}{\partial b_R} & \frac{\partial g_1}{\partial b_I} & \frac{\partial g_1}{\partial c_R} & \frac{\partial g_1}{\partial c_I} \\ \frac{\partial g_2}{\partial w_R} & \frac{\partial g_2}{\partial w_I} & \frac{\partial g_2}{\partial a_R} & \frac{\partial g_2}{\partial a_I} & \frac{\partial g_2}{\partial b_R} & \frac{\partial g_2}{\partial b_I} & \frac{\partial g_2}{\partial c_R} & \frac{\partial g_2}{\partial c_I} \end{pmatrix}.$$

The covariance matrix $V(\mathbf{\Gamma_0})$ expresses the uncertainty in the measured (corrected) VRC of the DUT.

7 Adaptive choice of calibration scheme

If m ($m > 3$) calibration standards are measured in a measurement cycle, this gives ${}^{m}C_3$ possible calibration schemes utilising three standards, ${}^{m}C_4$ possible over-determined calibration schemes utilising four standards and, in general, ${}^{m}C_n$ possible over-determined calibration schemes utilising n standards ($3 < n < m$) i.e. each time choosing n standards from m where the binomial coefficient is given by

$${}^{m}C_n = \frac{m!}{(m-n)!\, n!}\,.$$

As an alternative to employing all m standards in an over-determined calibration scheme, an adaptive choice of calibration scheme could be made. For each device and at each frequency, the uncertainty in VRC resulting from all of the possible calibration schemes (utilising 3, 4, …, m standards) could be calculated and a choice made of the scheme giving the lowest uncertainty. This form of adaptively choosing between candidate calibration schemes has been presented elsewhere [9].

8 Results obtained using over-determined VNA calibration schemes

In this section, we present some measurement results to illustrate the typical performance (in terms of achieved uncertainty of measurement[5]) of the VNA calibration schemes described in this paper. Figures 2 and 3 show the uncertainties (in real and imaginary components, respectively) of the VRC for a typical device (a mismatch termination with nominal VSWR of 1.1) measured using three different over-determined VNA calibration schemes. Also shown, for comparison purposes, are the uncertainties obtained using a short-open-load (SOL) calibration scheme where the standards have been characterised following the techniques given in section 5. The uncertainties quoted in the Figures and the Table are obtained as the square root of the diagonal elements of the covariance matrix $V(\mathbf{\Gamma_0})$ multiplied by a suitable coverage factor.

The over-determined calibration schemes use the same short-circuit, open-circuit and matched load from the SOL scheme together with one or two additional calibration standards. These are 'generalised' standards also characterised using the techniques in section 5. On this occasion, two additional 'mismatch' calibration standards[6] were used: the first, labelled MM1, consisted of a 3 dB attenuator terminated with an open-circuit; the second, labelled MM2, consisted of the same 3 dB attenuator, except this time terminated with a short-circuit. All measurements were made in coaxial line using devices fitted with precision 7 mm connectors.

The results in Figures 2 and 3 are further summarised in Table 1, which presents the uncertainty values achieved for the measurements at 500 MHz. It can be seen from these Figures and the Table that the use of additional calibration standards greatly reduces the overall uncertainty of measurement for the VNA, with the lowest uncertainties being obtained when all (five) standards are used.

Table 1: Mismatch termination (nominal VSWR = 1.1) at 500 MHz

Calibration scheme	Uncertainty in Real component of VRC	Uncertainty in Imaginary component of VRC
3 standards (SOL)	0.0024	0.0024
4 standards (SOL + MM1)	0.0018	0.0017
4 standards (SOL + MM2)	0.0016	0.0016
5 standards (SOL/MM1/MM2)	0.0014	0.0012

9 Conclusions

This paper has outlined an approach whereby the calibration of a VNA has been formulated in terms of a regression problem. In particular, it has been shown that Generalised Distance Regression provides a suitable representation of a VNA calibration and makes full use of all the uncertainty information (including correlation

[5] All uncertainties are stated at a level of confidence of 95% with traceability to the SI base units via primary national measurement standards (in this case, maintained by NPL).

[6] It should be noted that there is nothing particularly 'special' about these mismatch standards. The only design criterion with such standards is that they must provide a different value of VRC from other standards used during the calibration process. The mismatch standards used here have been chosen because they were very easy to produce (i.e. from readily available, off-the-shelf, components).

information represented by the appropriate terms in the covariance matrix) concerning the calibration standards. This information can be used subsequently to reduce the uncertainty (or, in other words, increase the accuracy) achieved by the VNA during measurement.

The presented approach has been illustrated using an example where five calibration standards have been made available. It has been shown that, on this occasion, using information from all five standards has greatly reduced the uncertainty of measurement achieved by the VNA. In fact, the results in Table 1 show a reduction in the uncertainty of approximately 50% compared with the conventional SOL calibration scheme, with values of uncertainty approaching ±0.001 (at a 95% level of confidence).

Finally, it is worth noting that, in principle, more than five standards (if available) could be used to perform the VNA calibration process. The results shown in this paper indicate that this could further reduce the uncertainty of measurement from that achieved using 'only' five standards. This could be an important consideration if, for example, this calibration technique were to be applied to devices such as contemporary electronic VNA calibrators, which are capable of generating many different 'standards' (i.e. VRC values) for use during the calibration process.

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11 References

[1] D Blackham, "Application of weighted least squares to OSL vector error correction", *61st ARFTG Conference Digest*, Philadelphia, PA, 13th June 2003.

[2] D F Williams, C M Wang and U Arz, "An optimal multiline TRL calibration algorithm", *2003 IEEE MTT-S Digest*, Philadelphia, PA, pp 1819-1822, June 2003.

[3] G F Engen and C A Hoer, "Thru-Reflect-Line: An improved technique for calibrating the dual six-port automatic network analyser", *IEEE Trans*, **MTT-27** (12), pp 987-993, December 1979.

[4] C A Hoer and G F Engen, "On-line accuracy assessment for the dual six-port ANA: extension to nonmating connectors", *IEEE Trans*, **IM-36** (2), pp 524-529, June 1987.

[5] N M Ridler, "A review of existing national measurement standards for RF and microwave impedance parameters in the UK", *IEE Colloquium Digest No 99/008*, pp 6/1-6/6, February 1999.

[6] N M Ridler, “News in RF impedance measurement”, *XXVIIth General Assembly of the International Union of Radio Science (URSI)*, paper no 437, session A1, Maastricht Exhibition and Congress Centre (MECC), The Netherlands, 17 August – 24 August 2002.

[7] M G Cox, M P Dainton and N M Ridler, “An interpolation scheme for precision reflection coefficient measurements at intermediate frequencies. Part 1: theoretical development”, *IMTC'2001 Proceedings of the 18th IEEE Instrumentation and Measurement Technology Conference*, pp 1720-1725, Budapest, Hungary, 21-23 May 2001.

[8] N M Ridler, M J Salter and P R Young, “An interpolation scheme for precision reflection coefficient measurements at intermediate frequencies. Part 2: practical implementation”, *IMTC'2001 Proceedings of the 18th IEEE Instrumentation and Measurement Technology Conference*, pp 1731-1735, Budapest, Hungary, 21-23 May 2001.

[9] A G Morgan, N M Ridler and M J Salter, “Generalised adaptive calibration schemes for RF network analysers based on minimising the uncertainty of measurement”, 60th *ARFTG Conference Digest*, Washington, DC, 5th and 6th December 2002.

[10] M G Cox, A B Forbes, P M Harris and I M Smith, “The classification and solution of regression problems for calibration”, *NPL Report CMSC 24/03*, May 2003.

[11] Glenn F Engen, “Microwave circuit theory and foundations of microwave metrology”, *IEE electrical measurement series*, Peter Peregrinus Ltd, Stevenage, 1992.

[12] G H Golub and C F Van Loan, “Matrix Computations”, John Hopkins University Press, Baltimore, third edition, 1996.

[13] M G Cox, M P Dainton, P M Harris, N M Ridler and P R Young, “A generalised treatment of the uncertainty in calibration and measurement of vector-indicating microwave reflectometers”, *NPL report CETM 10*, September 1999.

[14] N M Ridler and M J Salter, “Evaluating and expressing uncertainty in complex *S*-parameter measurements”, *56th ARFTG Conference Digest*, Boulder, CO, 30 November and 1 December 2000.

[15] N M Ridler and M J Salter, “Propagating *S*-parameter uncertainties to other measurement quantities”, *58th ARFTG Conference Digest*, San Diego, CA, 29th and 30th November 2001.

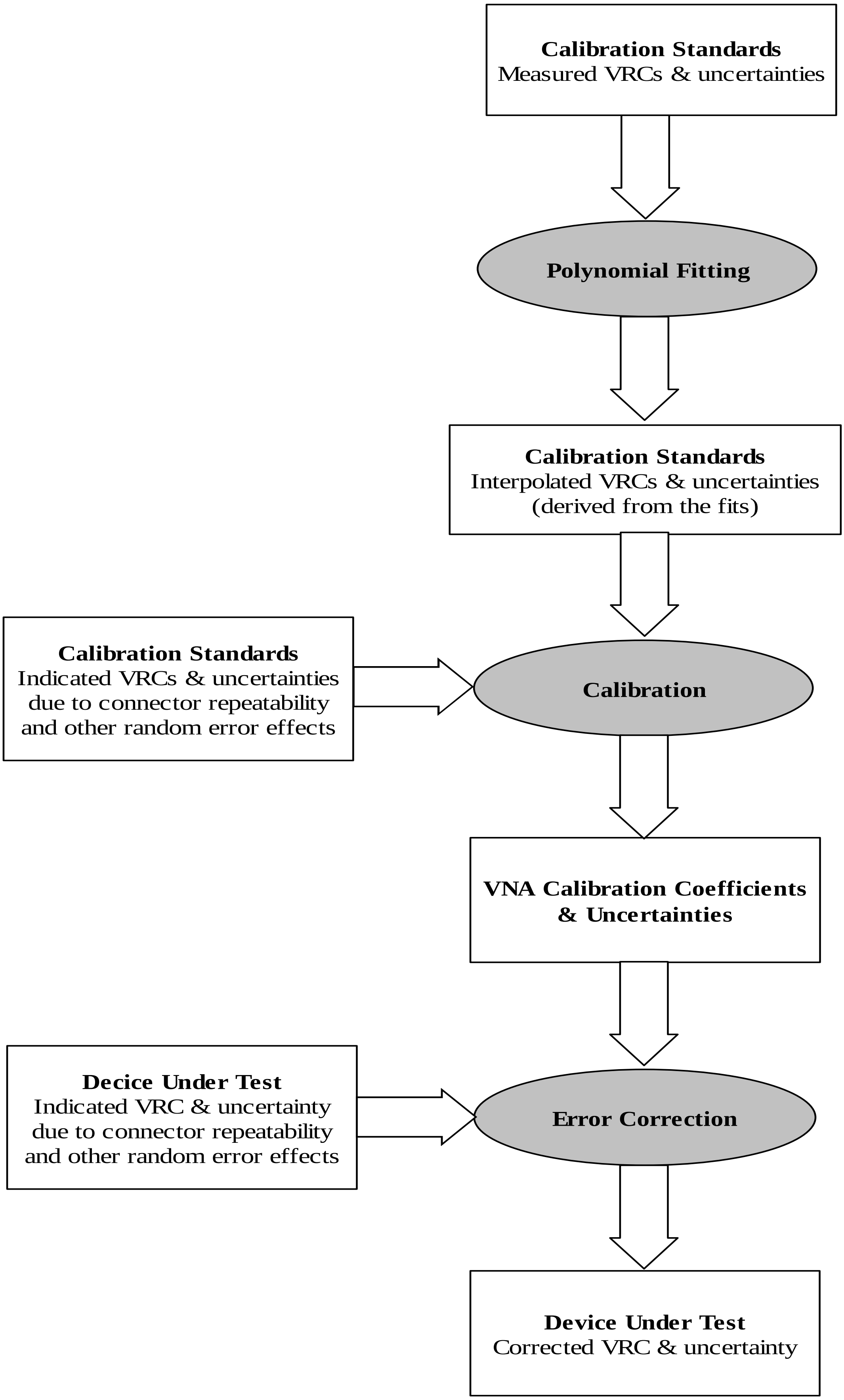

Figure 1: Flow graph showing the evaluation process for the uncertainty of measurement

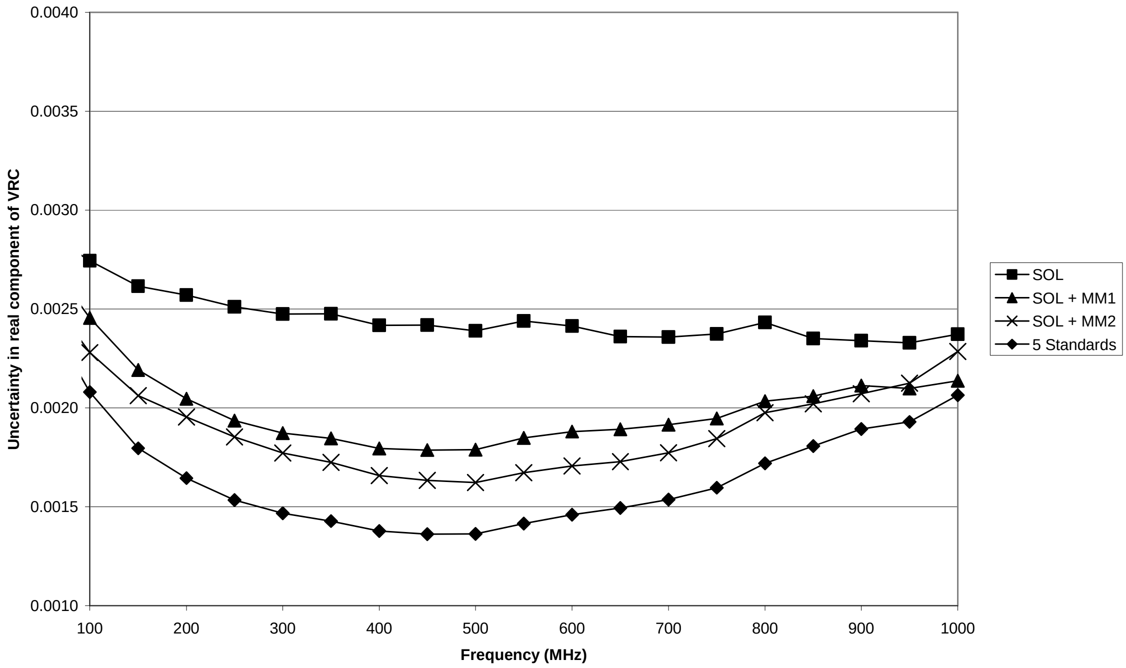

Figure 2: Uncertainty in the real component of the VRC of a mismatch termination (nominal VSWR = 1.1) achieved using four different calibration schemes

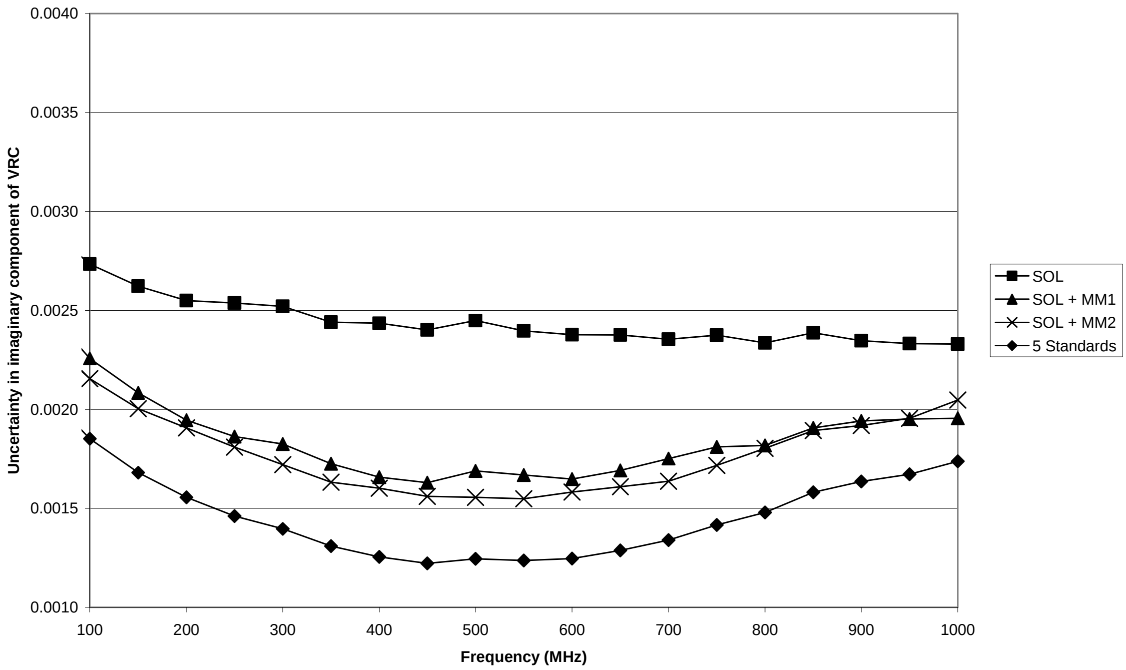

Figure 3: Uncertainty in the imaginary component of the VRC of a mismatch termination (nominal VSWR = 1.1) achieved using four different calibration schemes